KYUSHU-HET-109

Charged Lepton and Down-Type Quark Masses in $SU(1,1)$ Model and the Structure of Higgs Sector

Kenzo INOUE [*)] and Naoki YAMATSU[**)]

Department of Physics, Kyushu University, Fukuoka 812-8581, Japan

Abstract

The simplest noncompact group $SU(1,1)$, when introduced as a symmetry group of the generations of quarks and leptons in the framework of a vector-like theory, gives an excellent viewpoint on low energy physics. The minimal setup of the scheme, however, gives phenomenologically unacceptable prediction on the Yukawa coupling matrices. This suggests the higgs sector has richer structure than we expect from the success of MSSM. The natural extension of the scheme, which has doubled structure in the higgs sector, is formulated. The framework admits this extension in a restrictive way. The possible patterns of Yukawa couplings are classified and the expressions of the coupling matrices are presented.

[*)] E-mail: inou1scp@mbox.nc.kyushu-u.ac.jp
[**)] E-mail: yamatsu@higgs.phys.kyushu-u.ac.jp

§1. Introduction

The most mysterious nature in low energy physics is a simple repetition of the three generations of quarks and leptons. Nonetheless, they have a remarkable mass structure, the inter-generation mass hierarchy. Such a subtlety should have its origin in a definite principle in Nature. Thus, it will be legitimate to search for the principle based on the symmetry which governs the generations, that is, horizontal symmetry G_H.[1),2)]

One of the possible models was proposed based on the noncompact gauge symmetry $G_H = SU(1,1)$.[3)] This model is a vector-like[4)] realization[5)] of the minimal supersymmetric standard model (MSSM).[6),7)] The model contains minimal numbers of vector-like matter multiplets F and $\bar{F}$, which belong to infinite dimensional unitary representations of $SU(1,1)$. The appearance of three chiral generations results from the spontaneous breakdown of $SU(1,1)$. What is more, this symmetry breaking naturally realizes the hierarchy in the Yukawa couplings of higgses to quarks and leptons based on the group theoretical structure of $SU(1,1)$.

Since the original $SU(1,1)$ model was proposed, some attempts have been made[8),9)] to examine the phenomenological feasibility of the model, and also to inquire into the further potentialities of the model. Through these analyses, it was shown that this model reproduces, even under the simplest choice of the parameters, the quark and lepton mass hierarchies at least qualitatively. But quantitatively, it was not able to exactly do so for down-type quarks and charged leptons simultaneously, because their observed mass structures are somewhat different and the simplest choice of the parameters cannot produce this difference. One may imagine the relaxation of the restriction on the parameters solves this difficulty. We have examined it and found that the minimal setup of the model gives definite prediction on the mass ratios of charged leptons and down-type quarks which is phenomenologically unacceptable for any value of the parameters.

In this paper, we first clarify the reason why the minimal $SU(1,1)$ model does not reproduce both of the mass hierarchies. As we will see, if we introduce, according to the success of MSSM,[10)] only one down-type higgs multiplet, the structure of its Yukawa couplings to down-type quarks and charged leptons is essentially controlled by the $SU(1,1)$ group property, which turns out to be too rigid to compromise by the remaining freedom of the model. This suggests that the acceptable model should have at least doubled higgs structure. In order to reproduce MSSM at low energy, they must mix each other, and realize only one combination as a chiral down-type higgs. The main purpose of this paper is to examine the possible extension of the higgs sector within the basic framework of the model to have in some extent the flexibility for the structure of Yukawa couplings.

	Q	$\bar{U}$	$\bar{D}$	L	$\bar{E}$	H	H'	Ψ's
$SU(1,1)$	$+\alpha$	$+\beta$	$+\gamma$	$+\eta$	$+\lambda$	$-\rho$	$-\sigma$	finite
$SU_3 \times SU_2$	$(3,2)$	$(3^*,1)$	$(3^*,1)$	$(1,2)$	$(1,1)$	$(1,2)$	$(1,2)$	$(1,1)$
$Y/2$	$+1/6$	$-2/3$	$1/3$	$-1/2$	$+1$	$+1/2$	$-1/2$	0

Table I. The $SU_3 \times SU_2 \times U_1 \times SU(1,1)$ assignment for the multiplets in the $SU(1,1)$ model

In section 2, we give a setup of the minimal $SU(1,1)$ model for the fixing of the notation. In section 3, we discuss a property of the minimal model and show the reason why this model fails. In section 4, we examine the extension of the higgs sector and formulate the novel mixing scheme. We classify the possible patterns of mixing, which are related to the allowed patterns of the Yukawa couplings. The resulting structures of the Yukawa couplings are given in section 5. Section 6 is devoted to the summary and discussions.

§2. SU(1,1) model

Let Q_α be a multiplet carrying the $SU_3 \times SU_2 \times U_1$ quantum numbers of quark doublet q and belonging to an infinite-dimensional unitary representation of $SU(1,1)$ with a positive lowest weight α. Its conjugate $\bar{Q}_{-\alpha}$ has a negative highest weight $-\alpha$;

$$Q_\alpha = \{q_\alpha, q_{\alpha+1}, \cdots\}, \qquad \bar{Q}_{-\alpha} = \{\bar{q}_{-\alpha}, \bar{q}_{-\alpha-1}, \cdots\}. \tag{2·1}$$

The MSSM superfields q, $\bar{u}$, $\bar{d}$, ℓ, $\bar{e}$, h, h' are embedded into

$$Q_\alpha,\ \bar{U}_\beta,\ \bar{D}_\gamma,\ L_\eta,\ \bar{E}_\lambda,\ H_{-\rho},\ H'_{-\sigma}, \tag{2·2}$$

respectively. For the vector-like nature of the model, we also have the conjugate multiplets

$$\bar{Q}_{-\alpha},\ U_{-\beta},\ D_{-\gamma},\ \bar{L}_{-\eta},\ E_{-\lambda},\ \bar{H}_\rho,\ \bar{H}'_\sigma. \tag{2·3}$$

In addition to (2·2) and (2·3), we need some set of finite-dimensional non-unitary multiplets Ψ's of a type

$$\Psi = \{\psi_{-S}, \psi_{-S+1}, \cdots, \psi_{S-1}, \psi_S\}. \tag{2·4}$$

They are $SU_3 \times SU_2 \times U_1$ singlets, and are responsible to the spontaneous breakdown of $SU(1,1)$. They are indispensable to realize the chiral world at low energy from originally vector-like theory. For example, the coupling $Q\bar{Q}\Psi^F$ with the vacuum expectation value (VEV) $\langle\psi^F_{-3}\rangle$ of Ψ^F generates three generations of chiral quark doublets $q_m \equiv q_{\alpha+m}$ ($m = 0,1,2$), because q_m's disappear from the mass operator $Q\bar{Q}\langle\Psi^F\rangle$ owing to the weight conservation. The other components of Q form with the components of $\bar{Q}$ the huge Dirac mass

terms $\sum_{r=0}^{\infty} M_r\, q_{\alpha+3+r}\, \bar{q}_{-\alpha-r}$ with mass M_r blowing up in the limit $r \to \infty$ as $M_r \propto \langle \psi^F_{-3} \rangle\, r^{S^F}$ by the highest weight S^F of Ψ^F.[3)] Therefore, the superpotential

$$W_0 = Q\bar{Q}\Psi^F + \bar{U}U\Psi^F + \bar{D}D\Psi^F + L\bar{L}\Psi^F + \bar{E}E\Psi^F + H\bar{H}(\Psi^0 + \Psi^1) + H'\bar{H}'(\Psi'^0 + \Psi'^1) \quad (2\cdot5)$$

generates three chiral generations of q, $\bar{u}$, $\bar{d}$, ℓ, $\bar{e}$ through the VEV $\langle \psi^F_{-3} \rangle$. The VEVs $\langle \psi^0_0 \rangle$,$\langle \psi^1_1 \rangle$ of Ψ^0,Ψ^1 and $\langle \psi'^0_0 \rangle$,$\langle \psi'^1_1 \rangle$ of Ψ'^0,Ψ'^1 generate one generation of chiral higgs doublets h and h' as linear combinations of infinite components of H and H', respectively. We assume all VEVs $\langle \psi \rangle$'s are roughly of order $M \simeq 10^{16}$GeV to reproduce MSSM.

In this paper, we use the phase convention of the $SU(1,1)$ multiplets F_α, $\bar{G}_{-\alpha}$ and Ψ so that the bilinears

$$\sum_{n=0}^{\infty} (-1)^n f_{\alpha+n} \bar{g}_{-\alpha-n}, \quad \sum_{n=0}^{2S} (-1)^n \psi^*_{-S+n} \psi_{-S+n}, \quad \sum_{n=0}^{2S} (-1)^n \psi_{S-n} \psi_{-S+n} \qquad (2\cdot6)$$

are $SU(1,1)$ invariants. The $SU_3 \times SU_2 \times U_1 \times SU(1,1)$ assignment for the multiplets is shown in Table I.

The most general cubic superpotential of the multiplets in (2·2) and (2·3) that is compatible with the $SU_3 \times SU_2 \times U_1 \times SU(1,1)$ invariance is

$$\begin{aligned} W_1 = {} & \bar{U}QH + \bar{D}QH' + \bar{E}LH' + U\bar{Q}\bar{H} + D\bar{Q}\bar{H}' + E\bar{L}\bar{H}' \\ & + QQD + Q\bar{U}\bar{L} + \bar{Q}\bar{Q}\bar{D} + \bar{Q}UL, \end{aligned} \qquad (2\cdot7)$$

where we have abbreviated the coupling constant of each operator. The first line of (2·7) consists of the ordinary Yukawa couplings and their "mirror couplings". The second line contains operators which violate baryon-number and lepton-number conservations. Their low energy effects, however, are suppressed by the huge mass M. The dangerous dimension-4 ($QL\bar{D}, \bar{D}\bar{D}\bar{U}, LL\bar{E}$) and dimension-5 ($\bar{U}\bar{U}\bar{D}\bar{E}$) operators that embarrass MSSM[11)] are all forbidden by the $SU(1,1)$ symmetry because all of them have positive weights. The coupling $\bar{E}H'H'$ is incompatible with $\bar{E}LH'$ in the weight constraint given below.

The outstanding property of the scheme is that all coupling constants in (2·5) and (2·7) can be taken to be real under suitable phase convention of each multiplet. This allows us to settle the invariance under space-inversion (P), charge-conjugation (C) and time-reversal (T) as a "fundamental principle" of Nature. All of their violations observed in low energy physics are attributed to the spontaneous breakdown of $SU(1,1)$.

The $SU(1,1)$ invariance gives a rigid constraint to the couplings in (2·7). For example, the coupling $\bar{U}QH$ is allowed only when the weights of each multiplet satisfy $\Delta \equiv \rho - \alpha - \beta =$ [non-negative integer]. It has been shown[3)] that the simple lowest coupling ($\Delta = 0$) is

indispensable for the operators in the first line of (2·7) to realize a Yukawa coupling hierarchy. This gives the restrictions

$$\alpha+\beta=\rho, \qquad \alpha+\gamma=\eta+\lambda=\sigma. \tag{2·8}$$

The low energy manifestation of the minimal model is the MSSM multiplets q_m, $\bar{u}_m$, $\bar{d}_m$, ℓ_m, $\bar{e}_m$ $(m=0,1,2)$, h, h', that couple through the effective superpotential

$$W_{\rm eff}=\sum_{m,n=0}^{2}\left(y_u^{mn}\bar{u}_m q_n h+y_d^{mn}\bar{d}_m q_n h'+y_e^{mn}\bar{e}_m\ell_n h'\right). \tag{2·9}$$

We will not discuss in this paper on the mass structure of the up-type quarks related to the up-type higgs h, that has been nicely reproduced within the minimal setup of the model.[8] For the neutrino masses,[12),13)] which are also related to h, the adequate extension of the scheme that generates the effective operator $\kappa_\nu^{mn}\ell_m h\ell_n h$ has been realized.[9] For a reliable discussion of the CKM matrix[14] and the MNS matrix,[15] we need to clarify the both structures of up-type and down-type higgs sectors. Thus, we concentrate, in this paper, on the down-type higgs sector, which has been much problematic.

§3. Aspect of minimal model

Let us start by giving the embedding of the MSSM chiral multiplets into the SU(1,1) multiplets;

$$\begin{aligned}
q_{\alpha+i}&=\sum_{m=0}^{2}q_m U^q_{mi}+[\text{massive modes}], \quad \bar{u}_{\beta+i}=\sum_{m=0}^{2}\bar{u}_m U^u_{mi}+[\text{massive modes}],\\
\bar{d}_{\beta+i}&=\sum_{m=0}^{2}\bar{d}_m U^d_{mi}+[\text{massive modes}],\\
\ell_{\eta+i}&=\sum_{m=0}^{2}\ell_m U^\ell_{mi}+[\text{massive modes}], \quad \bar{e}_{\lambda+i}=\sum_{m=0}^{2}\bar{e}_m U^e_{mi}+[\text{massive modes}],\\
h_{-\rho-i}&=hU_i+[\text{massive modes}], \qquad h'_{-\sigma-i}=h'U'_i+[\text{massive modes}].
\end{aligned} \tag{3·1}$$

Since $Q,\bar{U},\bar{D},L,\bar{E},H,H'$ are unitary representations of SU(1,1), all coefficients U_i's should be the row vectors in the unitary matrices. Thus, they satisfy

$$\sum_{i=0}^{\infty}U^{q*}_{mi}U^q_{ni}=\delta_{mn}, \quad \text{etc.} \tag{3·2}$$

The Yukawa coupling of down-type higgs h' to leptons is derived by extracting the massless modes from the $SU(1,1)$ invariant coupling

$$y_E\bar{E}LH'=y_E\sum_{i,j=0}^{\infty}C^E_{i,j}\bar{e}_{\lambda+i}\ell_{\eta+j}h'_{-\sigma-i-j}\to\sum_{m,n=0}^{2}y_e^{mn}\bar{e}_m\ell_n h', \tag{3·3}$$

where $C^E_{i,j}$ is the Clebsch-Gordan(C-G) coefficient. Therefore, the coupling matrix y_e^{mn} has a general form

$$y_e^{mn} = y_E \sum_{i,j=0}^{\infty} C^E_{i,j} U^e_{mi} U^\ell_{nj} U'_{i+j}, \qquad m,n = 0,1,2. \tag{3·4}$$

For the C-G coefficient $C^E_{i,j}$, we give here, for the later convenience, the general expression which covers the non-lowest coupling ($\varDelta \geq 0$)

$$\bar{E}_\lambda L_\eta H'_{-\sigma-\varDelta} = \sum_{i,j=0}^{\infty} C^{\lambda,\eta}_{i,j}(\varDelta)\ \bar{e}_{\lambda+i}\ell_{\eta+j}h'_{-(\sigma+\varDelta)-i-j+\varDelta}, \tag{3·5}$$

where $\sigma = \lambda + \eta$. We note $C^{\lambda,\eta}_{i,j}(\varDelta) = 0$ for $i+j < \varDelta$. When $i + j \geq \varDelta$, it is given by

$$\begin{aligned} C^{\lambda,\eta}_{i,j}(\varDelta) = (-1)^{i+j}&\sqrt{\frac{i!j!\varGamma(2\lambda+i)\varGamma(2\eta+j)}{(i+j-\varDelta)!\varGamma(2\sigma+i+j+\varDelta)}} \\ &\times \sum_{r=0}^{\varDelta}(-1)^r \frac{(i+j-\varDelta)!\varGamma(2\lambda)\varGamma(2\eta)}{(i-r)!(j+r-\varDelta)!r!(\varDelta-r)!\varGamma(2\lambda+r)\varGamma(2\eta-r+\varDelta)}, \end{aligned} \tag{3·6}$$

which satisfies the symmetry relation $C^{\lambda,\eta}_{i,j}(\varDelta) = (-1)^\varDelta C^{\eta,\lambda}_{j,i}(\varDelta)$. Thus, we have

$$C^E_{i,j} = C^{\lambda,\eta}_{i,j}(0) = (-1)^{i+j}\sqrt{\frac{(i+j)!\varGamma(2\lambda+i)\varGamma(2\eta+j)}{i!j!\varGamma(2\sigma+i+j)}}. \tag{3·7}$$

It is instructive to examine a trivial case in (3·4) where the higgs doublet h' is realized as a pure $i = 0$ component of $h'_{-\sigma-i}$, that is, $U'_i = \delta_{i0}$. This gives

$$y_e^{mn} = y_E C^E_{0,0} U^e_{m0} U^\ell_{n0}. \tag{3·8}$$

Since the rank of this matrix is 1, only one generation of leptons, expressed as $\ell = \sum_{n=0}^{2} \ell_n U^\ell_{n0}$ and $\bar{e} = \sum_{m=0}^{2} \bar{e}_m U^e_{m0}$, has Yukawa coupling to h', and remaining two orthogonal generations decouple from h'. Therefore, it is indispensable to introduce the mixing in the realization of h'.

Our basic ansatz is to give the mixing coefficients U'_i of h' the "hierarchical" structure of the form

$$U'_i = U'_0(-\epsilon')^i b_i(\sigma), \qquad \epsilon' \lesssim 1. \tag{3·9}$$

For quarks and leptons, we assume, for a while, three generations are realized through the VEV $\langle \psi^F_{-3} \rangle$ of $\varPsi^F$. This gives

$$U^\ell_{mi} = U^e_{mi} = \delta_{mi}, \qquad m = 0,1,2. \tag{3·10}$$

The coupling matrix y_e^{mn} is then given by

$$y_e^{mn} = y_E U_0' \epsilon'^{m+n} b_{m+n}(\sigma) \sqrt{\frac{(m+n)!\Gamma(2\lambda+m)\Gamma(2\eta+n)}{m!n!\Gamma(2\sigma+m+n)}}. \tag{3·11}$$

Let us show how the mixing coefficients U_i' are determined from the couplings $H'\bar{H}'(\Psi'^0 + \Psi'^1)$ in (2·5). The VEVs $\langle\psi_0'\rangle \equiv \langle\psi_0'^0\rangle$ and $\langle\psi_1'\rangle \equiv \langle\psi_1'^1\rangle$ produce the higgs mass operators

$$H'\bar{H}'\langle\Psi'^0\rangle + H'\bar{H}'\langle\Psi'^1\rangle = \sum_{i=0}^{\infty} \left(C_i^{(0)} \langle\psi_0'\rangle h'_{-\sigma-i} \bar{h}'_{\sigma+i} + C_i^{(1)} \langle\psi_1'\rangle h'_{-\sigma-i-1} \bar{h}'_{\sigma+i} \right), \tag{3·12}$$

where $C_i^{(0)}$ and $C_i^{(1)}$ are the C-G coefficients. Since the massless mode h' is embedded in H' in the form $h'_{-\sigma-i} = U_i' h' + $ [massive modes], the orthogonality of h' to massive modes (the disappearance of h' from the mass operators (3·12)) requires the coefficients U_i' to satisfy the recursion equation

$$\langle\psi_0'\rangle C_i^{(0)} U_i' + \langle\psi_1'\rangle C_i^{(1)} U_{i+1}' = 0. \tag{3·13}$$

This equation precisely realizes the ansatz (3·9) of the mixing coefficients with

$$\epsilon' = \frac{\langle\psi_0'\rangle}{\langle\psi_1'\rangle}, \qquad b_i(\sigma) = \prod_{r=0}^{i-1} \frac{C_r^{(0)}}{C_r^{(1)}}. \tag{3·14}$$

For the relevant C-G coefficients, we first give the general formula for the coupling

$$H'_{-\sigma} \bar{K}'_\zeta \Psi' = \sum_{i,j=0}^{\infty} D_{i,j}^{\sigma,\zeta}(S)\ h'_{-\sigma-i} \bar{k}'_{\zeta+j} \psi'_{i-j+\bar{\Delta}}, \qquad \bar{\Delta} \equiv \sigma - \zeta, \tag{3·15}$$

which is allowed when the highest weight S of Ψ' satisfies $S \geq |\bar{\Delta}|$. For $-S \leq i - j + \bar{\Delta} \leq S$, we have

$$\begin{aligned} D_{i,j}^{\sigma,\zeta}(S) = (-1)^j & \sqrt{\frac{i!j!(i-j+S+\bar{\Delta})!(-i+j+S-\bar{\Delta})!}{\Gamma(2\sigma+i)\Gamma(2\zeta+j)}} \\ & \times \sum_{r=0}^{S-\bar{\Delta}} \frac{\Gamma(2\sigma+i+r)}{(S-\bar{\Delta}-r)!(-i+j+S-\bar{\Delta}-r)!r!(i-j+2\bar{\Delta}+r)!(i-S+\bar{\Delta}+r)!}, \end{aligned} \tag{3·16}$$

and otherwise $D_{i,j}^{\sigma,\zeta}(S) = 0$. Thus, $C_r^{(0)}$ and $C_r^{(1)}$ are given by

$$C_r^{(0)} = D_{r,r}^{\sigma,\sigma}(S_0), \qquad C_r^{(1)} = D_{r+1,r}^{\sigma,\sigma}(S_1), \tag{3·17}$$

where S_0 and S_1 are the highest weights of Ψ'^0 and Ψ'^1. For the lower values of S, they are

$$D_{r,r}^{\sigma,\sigma}(0) = (-1)^r, \quad D_{r,r}^{\sigma,\sigma}(1) = (-1)^r 2(\sigma+r), \quad D_{r+1,r}^{\sigma,\sigma}(1) = (-1)^r \sqrt{2(r+1)(2\sigma+r)}\,. \tag{3·18}$$

For the sake of convenience, let us define a normalized coupling matrix Y from (3·11) by

$$Y^{mn} = \frac{y_e^{mn}}{y_e^{00}} = \epsilon'^{m+n} \sqrt{\frac{(m+n)!}{m!n!} \frac{\Gamma(2\eta+n)\Gamma(2\lambda+m)\Gamma(2\sigma)}{\Gamma(2\eta)\Gamma(2\lambda)\Gamma(2\sigma+m+n)}}\; b_{m+n}(\sigma). \tag{3·19}$$

This matrix has a hierarchical structure

$$Y = \begin{pmatrix} 1 & \epsilon' \sqrt{\frac{\eta}{\sigma}}\; b_1(\sigma) & \epsilon'^2 \sqrt{\frac{(2\eta+1)\eta}{(2\sigma+1)\sigma}}\; b_2(\sigma) \\ \epsilon' \sqrt{\frac{\lambda}{\sigma}}\; b_1(\sigma) & \epsilon'^2 \sqrt{\frac{4\eta\lambda}{(2\sigma+1)\sigma}}\; b_2(\sigma) & \epsilon'^3 \sqrt{\frac{3(2\eta+1)\eta\lambda}{\sigma(\sigma+1)(2\sigma+1)}}\; b_3(\sigma) \\ \epsilon'^2 \sqrt{\frac{(2\lambda+1)\lambda}{(2\sigma+1)\sigma}}\; b_2(\sigma) & \epsilon'^3 \sqrt{\frac{3(2\lambda+1)\eta\lambda}{\sigma(\sigma+1)(2\sigma+1)}}\; b_3(\sigma) & \epsilon'^4 \sqrt{\frac{6(2\eta+1)(2\lambda+1)\eta\lambda}{\sigma(\sigma+1)(2\sigma+1)(2\sigma+3)}}\; b_4(\sigma) \end{pmatrix}. \tag{3·20}$$

When the eigenvalues Y_0, Y_1, Y_2 of this type of matrix are expanded in the power series of ϵ'^2 by

$$Y_0 = w_0 + O(\epsilon'^2), \quad Y_1 = \epsilon'^2 w_1 + O(\epsilon'^4), \quad Y_2 = \epsilon'^4 w_2 + O(\epsilon'^6), \tag{3·21}$$

the leading terms have the expressions

$$w_0 = Y^{00}, \tag{3·22}$$

$$\epsilon'^2 w_1 = \frac{Y^{00}Y^{11} - Y^{01}Y^{10}}{Y^{00}}, \tag{3·23}$$

$$\epsilon'^4 w_2 = \frac{Y^{22}(Y^{00}Y^{11} - Y^{01}Y^{10}) - Y^{21}(Y^{00}Y^{12} - Y^{10}Y^{02}) - Y^{20}(Y^{11}Y^{02} - Y^{01}Y^{12})}{Y^{00}Y^{11} - Y^{01}Y^{10}}. \tag{3·24}$$

The straightforward calculation gives the expressions of Y_0, Y_1, Y_2 in the form

$$Y_0 = 1 + O(\epsilon'^2), \tag{3·25}$$

$$Y_1 = \epsilon'^2 \sqrt{\frac{\eta\lambda}{(2\sigma+1)\sigma}}\; X(\sigma) + O(\epsilon'^4), \tag{3·26}$$

$$Y_2 = \epsilon'^4 \sqrt{\frac{6\eta\lambda(2\eta+1)(2\lambda+1)}{\sigma(\sigma+1)(2\sigma+1)(2\sigma+3)}}\; \frac{Z(\sigma)}{X(\sigma)} + O(\epsilon'^6), \tag{3·27}$$

where

$$X(\sigma) = 2b_2(\sigma) - b_1(\sigma)^2 \sqrt{\frac{2\sigma+1}{\sigma}}, \tag{3·28}$$

$$Z(\sigma) = 2b_2(\sigma)b_4(\sigma) + 2b_1(\sigma)b_2(\sigma)b_3(\sigma)\sqrt{\frac{2\sigma+3}{2\sigma}} \\ -3b_3(\sigma)^2 \sqrt{\frac{2\sigma+3}{6(\sigma+1)}} - 2b_2(\sigma)^3 \sqrt{\frac{(\sigma+1)(2\sigma+3)}{6\sigma(2\sigma+1)}} - b_1(\sigma)^2 b_4(\sigma)\sqrt{\frac{2\sigma+1}{\sigma}}. \tag{3·29}$$

The eigenvalues Y_0, Y_1, Y_2 are related to the charged lepton masses m_τ,m_μ,m_e through the ratios

$$R_e^0 \equiv \left|\frac{Y_1 Y_2}{Y_0^2}\right| = \frac{m_\mu m_e}{m_\tau^2}, \qquad R_e^1 \equiv \left|\frac{Y_0 Y_2}{Y_1^2}\right| = \frac{m_\tau m_e}{m_\mu^2}. \tag{3·30}$$

The ratio R_e^0 represents the magnitude of the mass hierarchy. The structure of the mass hierarchy is characterized by the ratio R_e^1. They are given by

$$R_e^0 = |\epsilon'|^6 \frac{\eta\lambda}{\sigma(2\sigma+1)}\sqrt{\frac{6(2\eta+1)(2\lambda+1)}{(\sigma+1)(2\sigma+3)}}\ |Z(\sigma)| + O(\epsilon'^8), \tag{3·31}$$

$$R_e^1 = \sqrt{\frac{6(2\eta+1)(2\lambda+1)\sigma(2\sigma+1)}{\eta\lambda(\sigma+1)(2\sigma+3)}}\ \left|\frac{Z(\sigma)}{X(\sigma)^3}\right| + O(\epsilon'^2). \tag{3·32}$$

These ratios should be compared with the observed values[16)]

$$R_e^0(\text{obs}) = \frac{m_\mu m_e}{m_\tau^2} \simeq 1.6\times 10^{-5}, \quad R_e^1(\text{obs}) = \frac{m_\tau m_e}{m_\mu^2} \simeq 0.082. \tag{3·33}$$

The remarkable feature of the $SU(1,1)$ model is that, although Y_1 and Y_2 are already suppressed by ϵ'^2 and ϵ'^4, respectively, when $\epsilon' < 1$, the factors from the C-G coefficients realize the unexpectedly large hierarchy through the strong cancellations within the terms in (3·28) and (3·29) owing to the fact that each element of Y in (3·20) is systematically controlled by the $SU(1,1)$ symmetry.

As an example, let us show this fact in the simplest case where $S_0 = 0$ and $S_1 = 1$. In this case, $b_n(\sigma)$ takes a form

$$b_n(\sigma) = \sqrt{\frac{\Gamma(2\sigma)}{2^n n!\,\Gamma(2\sigma+n)}}. \tag{3·34}$$

This gives

$$Y_1 = -\epsilon'^2 \frac{1}{2^2\sigma^2(2\sigma+1)}\sqrt{\eta\lambda} + O(\epsilon'^4), \tag{3·35}$$

$$Y_2 = \epsilon'^4 \frac{1}{2^4\sigma(\sigma+1)^2(2\sigma+1)^2(2\sigma+3)}\sqrt{\eta\lambda(2\eta+1)(2\lambda+1)} + O(\epsilon'^6), \tag{3·36}$$

and then

$$R_e^0 = |\epsilon'|^6 \frac{1}{2^6\sigma^3(\sigma+1)^2(2\sigma+1)^3(2\sigma+3)}\eta\lambda\sqrt{(2\eta+1)(2\lambda+1)} + O(\epsilon'^8), \tag{3·37}$$

$$R_e^1 = \frac{\sigma^3}{(\sigma+1)^2(2\sigma+3)}\sqrt{\frac{(2\eta+1)(2\lambda+1)}{\eta\lambda}} + O(\epsilon'^2). \tag{3·38}$$

If we simply set $2\eta = 2\lambda = \sigma = 1$, we have

$$R_e^0 = |\epsilon'|^6 \frac{1}{2^9 \cdot 3^3 \cdot 5} = |\epsilon'|^6 \cdot 1.4 \times 10^{-5} + O(\epsilon'^8), \qquad R_e^1 = 0.2 + O(\epsilon'^2). \tag{3·39}$$

Although the result for R_e^1 is unacceptable, this suggests that we need not introduce unnaturally small ϵ'. If we chose the weights as $2\eta = 2\lambda = \sigma = 1/2$, we obtain the reasonable values

$$R_e^0 = |\epsilon'|^6 \cdot 1.63 \times 10^{-4} + O(\epsilon'^8), \qquad R_e^1 = 0.083 + O(\epsilon'^2). \tag{3·40}$$

Fitting the observed mass hierarchy (3·33) by the leading term of (3·40), we obtain $\epsilon'^2 = 0.46$. When we diagonalize the matrix Y numerically with the input values $2\eta = 2\lambda = \sigma = 1/2,\ \epsilon'^2 = 0.46$, we find

$$R_e^0 = 1.1 \times 10^{-5}, \qquad R_e^1 = 0.092. \tag{3·41}$$

This means that the contribution from higher order terms is subdominant even when $|\epsilon'| \simeq 1$ and does not alter the value from the leading term by the factor beyond $O(1)$.

We may have an expectation, from this rough analysis, that this minimal $SU(1,1)$ model may give a hopeful scheme to understand the characteristic structure of the mass hierarchy of quarks and leptons. We will see, however, this scheme encounters a serious difficulty.

The down-type higgs doublet h' couples not only to charged leptons but also to down-type quarks through $SU(1,1)$ invariant couplings $y_E \bar{E} L H' + y_D \bar{D} Q H'$. Therefore, the quantities related to the latter are obtained from those of the former by simply replacing the $SU(1,1)$ weights η and λ by α and γ. Thus, from (3·31) and (3·32), we obtain the expressions of the mass ratios for down-type quarks;

$$R_d^0 = \frac{m_s m_d}{m_b^2} = |\epsilon'|^6 \frac{\alpha\gamma}{\sigma(2\sigma+1)} \sqrt{\frac{6(2\alpha+1)(2\gamma+1)}{(\sigma+1)(2\sigma+3)}}\, |Z(\sigma)| + O(\epsilon'^8), \tag{3·42}$$

$$R_d^1 = \frac{m_b m_d}{m_s^2} = \sqrt{\frac{6(2\alpha+1)(2\gamma+1)\sigma(2\sigma+1)}{\alpha\gamma(\sigma+1)(2\sigma+3)}} \left| \frac{Z(\sigma)}{X(\sigma)^3} \right| + O(\epsilon'^2). \tag{3·43}$$

We show the expressions (3·31), (3·32), (3·42), (3·43) are incompatible with the observation, when $O(\epsilon'^2)$ corrections are subdominant. To see this clearly, let us define the cross ratios $\mathcal{R}$ and $\mathcal{S}$ by

$$\mathcal{R} \equiv \left(\frac{m_s m_\tau}{m_b m_\mu}\right)^2, \qquad \mathcal{S} \equiv \left(\frac{m_d m_\mu}{m_s m_e}\right)^2. \tag{3·44}$$

Neglecting $O(\epsilon'^2)$ corrections, we find, for any functional form of $b_i(\sigma)$,

$$\mathcal{R} = \left(\frac{R_d^0 R_e^1}{R_e^0 R_d^1}\right)^{2/3} = \frac{\alpha\gamma}{\eta\lambda}, \qquad \mathcal{S} = \left(\frac{R_e^0}{R_d^0}\mathcal{R}\right)^{-2} = \frac{(2\alpha+1)(2\gamma+1)}{(2\eta+1)(2\lambda+1)}. \tag{3·45}$$

The restriction of the weights (2·8) combines these two expressions to

$$4(\mathcal{R}-\mathcal{S})\eta\lambda = (\mathcal{S}-1)(2\sigma+1), \tag{3·46}$$

which states, from the positivity of the weights, that $\mathcal{R}$ and $\mathcal{S}$ must satisfy

$$1 < \mathcal{S} < \mathcal{R} \quad \text{or} \quad \mathcal{R} < \mathcal{S} < 1. \tag{3·47}$$

This is the prediction of the minimal model. The observed values[16)]

$$\mathcal{R}(\text{obs}) = \left(\frac{m_s m_\tau}{m_b m_\mu}\right)^2 \simeq 0.1, \qquad \mathcal{S}(\text{obs}) = \left(\frac{m_d m_\mu}{m_s m_c}\right)^2 \simeq 100 \tag{3·48}$$

strongly conflict with the above prediction. The origin of this conflict is the observed values[16)]

$$R_d^0(\text{obs}) = \frac{m_s m_d}{m_b^2} \simeq 1\times 10^{-5}, \qquad R_d^1(\text{obs}) = \frac{m_b m_d}{m_s^2} \simeq 3, \tag{3·49}$$

which show, although the magnitudes of the hierarchy $R^0(\text{obs})$'s are in the same order, the structures of hierarchy $R_e^1(\text{obs}) \simeq 0.08$ and $R_d^1(\text{obs}) \simeq 3$ are too different to compromise by the allowed values of the weights and the possible higher order corrections of ϵ'^2. If we try to fit (3·49), we need to introduce the higher value of the highest weight S for Ψ'^0 and Ψ'^1, for example $S=3$.[8)] But in this case, we cannot fit $R_e^0(\text{obs})$ and $R_e^1(\text{obs})$.

One may imagine the difficulty is due to the simplest assumption (3·10), that identifies the first three components of the $SU(1,1)$ multiplets with the three chiral generations. In fact, we have a strong motivation,[8)] as will be explained at the end of section 4, to replace the couplings $Q\bar{Q}\Psi^F$, etc. in (2·5) by

$$(Q\bar{Q}+\bar{U}U+\bar{D}D+L\bar{L}+\bar{E}E)(\Psi^F+\Psi'^F) \tag{3·50}$$

with VEV $\langle\Psi'^F\rangle = \langle\psi_0'^F\rangle$ and the highest weight $S'^F = S^F \equiv S$. It is straightforward to confirm the mixing coefficients take a form

$$U_{nj}^\ell = U_n^\ell \sum_{s=0}^{\infty} \delta_{j,n+3s}(-\epsilon^F)^s b_{ns}^\ell(\eta), \qquad n=0,1,2 \tag{3·51}$$

with

$$\epsilon^F = \frac{\langle\psi_0'^F\rangle}{\langle\psi_{-3}^F\rangle}, \qquad b_{ns}^\ell(\eta) = \prod_{r=0}^{s-1} \frac{D_{3r+n,3r+n}^{\eta,\eta}(S)}{D_{3r+n,3r+n+3}^{\eta,\eta}(S)}. \tag{3·52}$$

The expression (3·4) then gives y_e^{mn} in the form

$$y_e^{mn} = y_E U_m^e U_n^\ell \sum_{r,s=0}^{\infty} (-\epsilon^F)^{r+s} C_{m+3r,n+3s}^E b_{mr}^e(\lambda) b_{ns}^\ell(\eta) U'_{m+n+3(r+s)}. \tag{3·53}$$

This clearly shows that the mixing effect $(r+s>0)$ is almost negligible.

§4. Extension of the higgs sector

The result of the previous section claims that we should change the basic setup of the model. It is now obvious that the difficulty is coming from the assumption that only one higgs multiplet H' supplies masses to both of charged leptons and down-type quarks. Thus, the legitimate remedy will be the extension of the higgs sector. We examine the minimal extension by doubling the higgs sector *a la* Georgi-Jarlskog.[17)]

We introduce, in addition to H' and $\bar{H}'$, another set of higgs multiplets K' and $\bar{K}'$ whose weights may not necessarily be equal to those of H' and $\bar{H}'$;

$$H'_{-\sigma}, \quad \bar{H}'_{\sigma}, \quad K'_{-\sigma-\Delta}, \quad \bar{K}'_{\sigma+\Delta}: \qquad \sigma+\Delta>0. \tag{4·1}$$

Since we wish to realize MSSM at low energy, we need a mixing scheme that generates only one chiral higgs doublet h' as a linear combination of the components of H' and K' in the way

$$h'_{-\sigma-n} = h'U'_n + [\text{massive modes}], \quad k'_{-\sigma-\Delta-n} = h'V'_n + [\text{massive modes}]. \tag{4·2}$$

For this purpose, we introduce finite-dimensional non-unitary $SU(1,1)$ multiplets $\Psi', \Phi', X', \Omega'$, and couple them to higgs multiplets by

$$W_{h'} = H'\bar{H}'\Psi' + K'\bar{K}'\Phi' + K'\bar{H}'X' + H'\bar{K}'\Omega'. \tag{4·3}$$

In order to keep the vector-like nature of the model, X' and Ω' must be assigned to the same representation of $SU(1,1)$. We already know the $SU(1,1)$ invariance requires the weights of Ψ' and Φ' to be integer. On the other hand, that of X' and Ω' is allowed to be integer (Type-I) or half-integer (Type-II).

Corresponding to the type of X' and Ω', the Yukawa couplings of H' and K' to leptons and quarks, which replace $\bar{D}QH' + \bar{E}LH'$ in (2·7), take different forms. The type-I case, which restricts Δ to be integer, allows H' and K' to couple to both of them by

$$\text{Type-I} : \qquad \bar{E}_\lambda L_\eta (H'_{-\sigma} + K'_{-\sigma-\Delta}) + \bar{D}_\gamma Q_\alpha (H'_{-\sigma} + K'_{-\sigma-\Delta}). \tag{4·4}$$

We assume H' takes the lowest coupling. Thus, we have

$$\lambda + \eta = \gamma + \alpha = \sigma, \qquad \Delta = 0, 1, 2, 3, \cdots. \tag{4·5}$$

On the other hand, the type-II case in (4·3) requires Δ to be half-integer. This forbids the simultaneous coupling of H' and K' like (4·4), and the Yukawa couplings are separated by

$$\text{Type-II} : \qquad \bar{E}_\lambda L_\eta H'_{-\sigma} + \bar{D}_\gamma Q_\alpha K'_{-\sigma-\Delta}. \tag{4·6}$$

In this case, both of H' and K' must take the lowest coupling. Thus, we have

$$\lambda + \eta = \sigma, \qquad \gamma + \alpha = \sigma + \mathit{\Delta}, \qquad \mathit{\Delta} = \mathit{\Delta}_{\min}, \cdots, -\frac{1}{2}, +\frac{1}{2}, +\frac{3}{2}, \cdots, \tag{4·7}$$

where $-\sigma < \mathit{\Delta}_{\min} \le -\sigma + 1$.

Now, let us proceed to the indispensable ingredient of the scheme, that is, how to generate one chiral higgs h' from doubled higgs sector through the superpotential (4·3). It will be natural to expect each finite-dimensional multiplet acquires non-vanishing VEV at its single component. So, we first arbitrarily assume the weights P, Q, M, and N of the VEVs by

$$\langle \mathit{\Psi}' \rangle = \langle \psi'_P \rangle, \quad \langle \mathit{\Phi}' \rangle = \langle \phi'_Q \rangle, \quad \langle X' \rangle = \langle \chi'_M \rangle, \quad \langle \mathit{\Omega}' \rangle = \langle \omega'_N \rangle. \tag{4·8}$$

Substituting the expressions (4·2) to the Higgs mass operators (4·3), we have

$$\begin{aligned} & H'\bar{H}'\langle \mathit{\Psi}' \rangle + K'\bar{K}'\langle \mathit{\Phi}' \rangle + K'\bar{H}'\langle X' \rangle + H'\bar{K}'\langle \mathit{\Omega}' \rangle \\ & = \sum_{n=0}^{\infty} h' \left(C_n^{(P)} \langle \psi'_P \rangle U'_{n+P} + F_n^{(M)} \langle \chi'_M \rangle V'_{n+M-\mathit{\Delta}} \right) \bar{h}'_{\sigma+n} \\ & \quad + \sum_{n=0}^{\infty} h' \left(D_n^{(Q)} \langle \phi'_Q \rangle V'_{n+Q} + F_n'^{(N)} \langle \omega'_N \rangle U'_{n+N+\mathit{\Delta}} \right) \bar{k}'_{\sigma+\mathit{\Delta}+n} + [\text{massive modes}] \,. \end{aligned} \tag{4·9}$$

All C-G coefficients are given in terms of the general formula (3·16) by

$$C_n^{(P)} = D^{\sigma,\ \sigma}_{n+P,\ n}(S_{\mathit{\Psi}}), \qquad D_n^{(Q)} = D^{\sigma+\mathit{\Delta},\ \sigma+\mathit{\Delta}}_{n+Q,\ n}(S_{\mathit{\Phi}}), \tag{4·10}$$

$$F_n^{(M)} = D^{\sigma+\mathit{\Delta},\ \sigma}_{n+M-\mathit{\Delta},\ n}(S_X), \quad F_n'^{(N)} = D^{\sigma,\ \sigma+\mathit{\Delta}}_{n+N+\mathit{\Delta},\ n}(S_{\mathit{\Omega}}), \qquad S_{\mathit{\Omega}} = S_X. \tag{4·11}$$

The disappearance of h' from (4·9) requires the coefficients of $\bar{h}'_{\sigma+n}$ and $\bar{k}'_{\sigma+\mathit{\Delta}+n}$ to vanish together;

$$C_n^{(P)} \langle \psi'_P \rangle U'_{n+P} + F_n^{(M)} \langle \chi'_M \rangle V'_{n+M-\mathit{\Delta}} = 0, \tag{4·12}$$

$$D_n^{(Q)} \langle \phi'_Q \rangle V'_{n+Q} + F_n'^{(N)} \langle \omega'_N \rangle U'_{n+N+\mathit{\Delta}} = 0. \tag{4·13}$$

If we wish to have only one chiral h', these recursion equations must determine all elements of U'_n and V'_n uniquely by the single input value U'_0 or V'_0. This is possible only when just one among the four numbers P, Q, $M - \mathit{\Delta}$, and $N + \mathit{\Delta}$ appearing in the subscripts of U'_i and V'_i is 1 and others are 0.

Therefore, we have four cases that realize single chiral h';

$$\text{case A}: \quad P = 1, \quad Q = 0, \quad M = \mathit{\Delta}, \qquad N = -\mathit{\Delta}, \tag{4·14}$$

$$\text{case B}: \quad P = 0, \quad Q = 1, \quad M = \mathit{\Delta}, \qquad N = -\mathit{\Delta}, \tag{4·15}$$

$$\text{case C}: \quad P = 0, \quad Q = 0, \quad M = \mathit{\Delta} + 1, \quad N = -\mathit{\Delta}, \tag{4·16}$$

$$\text{case D}: \quad P = 0, \quad Q = 0, \quad M = \mathit{\Delta}, \qquad N = 1 - \mathit{\Delta}. \tag{4·17}$$

The mixing coefficients U'_n and V'_n then satisfy the recursion equations

$$\text{case A}: \quad U'_{n+1} = \epsilon' \frac{F_n^{(\Delta)} F_n'^{(-\Delta)}}{C_n^{(1)} D_n^{(0)}} U'_n, \qquad V'_{n+1} = \epsilon' \frac{F_n^{(\Delta)} F_{n+1}'^{(-\Delta)}}{C_n^{(1)} D_{n+1}^{(0)}} V'_n, \tag{4·18}$$

$$\text{case B}: \quad U'_{n+1} = \epsilon' \frac{F_{n+1}^{(\Delta)} F_n'^{(-\Delta)}}{C_{n+1}^{(0)} D_n^{(1)}} U'_n, \qquad V'_{n+1} = \epsilon' \frac{F_n^{(\Delta)} F_n'^{(-\Delta)}}{C_n^{(0)} D_n^{(1)}} V'_n, \tag{4·19}$$

$$\text{case C}: \quad U'_{n+1} = \epsilon' \frac{C_n^{(0)} D_{n+1}^{(0)}}{F_n^{(\Delta+1)} F_{n+1}'^{(-\Delta)}} U'_n, \quad V'_{n+1} = \epsilon' \frac{C_n^{(0)} D_n^{(0)}}{F_n^{(\Delta+1)} F_n'^{(-\Delta)}} V'_n, \tag{4·20}$$

$$\text{case D}: \quad U'_{n+1} = \epsilon' \frac{C_n^{(0)} D_n^{(0)}}{F_n^{(\Delta)} F_n'^{(1-\Delta)}} U'_n, \qquad V'_{n+1} = \epsilon' \frac{C_{n+1}^{(0)} D_n^{(0)}}{F_{n+1}^{(\Delta)} F_n'^{(1-\Delta)}} V'_n, \tag{4·21}$$

where ϵ' is given for each case by

$$\text{case A, B}: \quad \epsilon' \equiv \frac{\langle \chi'_M \rangle \langle \omega'_N \rangle}{\langle \psi'_P \rangle \langle \phi'_Q \rangle}, \qquad \text{case C, D}: \quad \epsilon' \equiv \frac{\langle \psi'_P \rangle \langle \phi'_Q \rangle}{\langle \chi'_M \rangle \langle \omega'_N \rangle}, \tag{4·22}$$

and input values U'_0 and V'_0 are connected by (4·13) for case A and C and by (4·12) for case B and D. We take a phase convention of h' so that U'_0 is real and positive.

One may suspect that the couplings (4·3) generate, in addition to h', chiral $\bar{h}'$ in $\bar{H}'$ and $\bar{K}'$ in the form

$$\bar{h}'_{\sigma+n} = \bar{h}' \bar{U}'_n + [\text{massive modes}], \quad \bar{k}'_{\sigma+\Delta+n} = \bar{h}' \bar{V}'_n + [\text{massive modes}]. \tag{4·23}$$

It is straightforward to confirm that the set of VEVs of all four cases (4·14)∼(4·17) gives $\bar{U}'_n = \bar{V}'_n = 0$.

The pattern of the VEVs (4·14)∼(4·17) is necessary for an appearance of single h'. But this is not sufficient. The realization of h' further requires the normalizable conditions $\sum_{n=0}^{\infty} |U'_n|^2 < \infty$ and $\sum_{n=0}^{\infty} |V'_n|^2 < \infty$. If these conditions were not satisfied, h' were an illusion without any physical reality. That is, we need to have

$$\lim_{n\to\infty} \left| \frac{U'_{n+1}}{U'_n} \right| < 1, \qquad \lim_{n\to\infty} \left| \frac{V'_{n+1}}{V'_n} \right| < 1. \tag{4·24}$$

This requires the knowledge of the asymptotic behavior of $D_{i,j}^{\sigma,\zeta}(S)$ in the limit $i, j \to \infty$ with $|i-j|$ fixed. The expression (3·16) gives

$$D_{i,j}^{\sigma,\zeta}(S) \simeq (-1)^j j^S \frac{(2S)!}{(S-\bar{\Delta})!(S+\bar{\Delta})!\sqrt{(S-\bar{\Delta}-i+j)!(S+\bar{\Delta}+i-j)!}}. \tag{4·25}$$

From this asymptotic behavior, we find

$$\text{case A, B}: \quad \lim_{n\to\infty} \left| \frac{U'_{n+1}}{U'_n} \right| = \lim_{n\to\infty} \left| \frac{V'_{n+1}}{V'_n} \right| = \frac{|\epsilon'|}{L} n^{2S_X - S_\Psi - S_\Phi}, \tag{4·26}$$

$$\text{case C, D}: \quad \lim_{n\to\infty} \left| \frac{U'_{n+1}}{U'_n} \right| = \lim_{n\to\infty} \left| \frac{V'_{n+1}}{V'_n} \right| = \frac{|\epsilon'|}{L} n^{S_\Psi + S_\Phi - 2S_X}, \tag{4·27}$$

with case-dependent but n-independent number L. This result shows that chiral h' appears for any finite value of ϵ' when the highest weights of the multiplets Ψ', Φ', X', and Ω' satisfy

$$\text{case A, B}: \quad S_\Psi + S_\Phi > 2S_X, \qquad \text{case C, D}: \quad 2S_X > S_\Psi + S_\Phi. \tag{4·28}$$

In these cases, the hierarchy between U'_n and U'_{n+1} (and also between V'_n and V'_{n+1}) becomes larger with the increase of n.

The situation that faithfully realizes the basic ansatz $U'_n \propto \epsilon'^n$, $V'_n \propto \epsilon'^n$ is a marginal case;

$$S_\Psi + S_\Phi = 2S_X. \tag{4·29}$$

In this case, ϵ' must be subject to the constraint

$$|\epsilon'| < L. \tag{4·30}$$

There is a convincing reason to prefer the marginal assignment (4·29).[8] It is sensible to expect there are plenty of particles in Nature more than those of MSSM. Some of them may be related to H' and K' by some symmetry G to form irreducible representations of G. As an illustration, let us consider the grand unified theory (GUT) with $G = SU(5)$.[19),20)] In this case, we inevitably have color-triplet GUT partners H'_c and K'_c that form **5**'s of $SU(5)$ with H' and K'. The non-marginal assignment (4·28) generates, in addition to h', color-triplet chiral h'_c in the low energy. The marginal assignment solves this notorious doublet-triplet mass problem in a natural way. Suppose X' and Ω' belong to **1**, and Ψ' and Φ' to **24** of $SU(5)$ in case A. The difference of the $SU(5)$ C-G coefficients of **24** for H' and H'_c $(1 : -2/3)$ brings the difference of L for h' and L_c for h'_c by $L_c = (-2/3)^2 L$. This means that, when ϵ' takes a value so that $\frac{4}{9}L < |\epsilon'| < L$, h'_c becomes an illusion and disappears from physical world but h' survives as a chiral superfield.

§5. Details of the Yukawa couplings

Now, let us discuss the detailed structure of the Yukawa couplings in Type-I and Type-II schemes. Since we have no clue at hand on the dynamics to which the non-unitary finite-dimensional multiplets should be subject, it will be fair to treat all four patterns of VEVs (4·14)∼(4·17) as the possible candidates. We omit the mixing effect of quarks and leptons that has been shown to be negligible.

Type-I scheme

The Yukawa coupling of h' to leptons in the Type-I scheme (4·4) is now

$$y_E \bar{E} L H' + y_E^\Delta \bar{E} L K'$$

$$
\begin{aligned}
&= \sum_{i,j=0}^{\infty} \left(y_E C^E_{i,j} \bar{e}_{\lambda+i} l_{\eta+j} h'_{-\sigma-i-j} + y^{\Delta}_E \sum_{k=0}^{\infty} C^{E\Delta}_{i,j} \bar{e}_{\lambda+i} l_{\eta+j} k'_{-\sigma-k-\Delta} \delta_{k,i+j-\Delta} \right) \\
&\longrightarrow \sum_{m,n=0}^{2} (y_E C^E_{m,n} U'_{m+n} + y^{\Delta}_E C^{E\Delta}_{m,n} V'_{m+n-\Delta}) \bar{e}_m l_n h',
\end{aligned}
\tag{5·1}
$$

that is,

$$
y^{mn}_e = y_E C^E_{m,n} U'_{m+n} + y^{\Delta}_E C^{E\Delta}_{m,n} V'_{m+n-\Delta}. \tag{5·2}
$$

The C-G coefficient $C^E_{m,n}$ and $C^{E\Delta}_{m,n}$ are expressed in terms of $C^{\lambda,\eta}_{i,j}(\Delta)$ given in (3·6) by

$$
C^E_{m,n} = C^{\lambda,\eta}_{m,n}(0), \qquad C^{E\Delta}_{m,n} = C^{\lambda,\eta}_{m,n}(\Delta). \tag{5·3}
$$

Note that $V'_{n<0} = 0$. Therefore, the second term in (5·2) contributes only to the matrix elements with $m+n \geq \Delta$. Because U'_n and V'_n are definitely related by (4·12) and (4·13), the coupling matrix y^{mn}_e in the Type-I scheme is generally represented in the form

$$
y^{mn}_e = y_E C^E_{m,n} \left(1 - r_E \theta_{m+n,\Delta} \frac{C^{E\Delta}_{m,n}}{C^E_{m,n}} \Pi_{m+n}(\sigma) \right) U'_{m+n}, \tag{5·4}
$$

where $\theta_{m+n,\Delta} = 1$ for $m+m \geq \Delta$ and 0 for $m+n < \Delta$. To derive the expressions for the numerical factor r_E and the function $\Pi_{m+n}(\sigma)$, first perform Δ steps of recursion from $V'_{m-\Delta}$ to V'_m by (4·18)∼(4·21), and rewrite V'_m in terms of U'_m by (4·13) for case A and C and by (4·12) for case B and D. The result is

$$
\text{case A}: \quad r_E = \frac{y^{\Delta}_E}{y_E} \frac{\langle \omega'_{-\Delta} \rangle}{\langle \phi'_0 \rangle} (\epsilon')^{-\Delta}, \qquad \Pi_m(\sigma) = \frac{F'^{(-\Delta)}_m}{D^{(0)}_m} \prod_{r=1}^{\Delta} \frac{C^{(1)}_{m-r} D^{(0)}_{m-r+1}}{F^{(\Delta)}_{m-r} F'^{(-\Delta)}_{m-r+1}}, \tag{5·5}
$$

$$
\text{case B}: \quad r_E = \frac{y^{\Delta}_E}{y_E} \frac{\langle \psi'_0 \rangle}{\langle \chi'_{\Delta} \rangle} (\epsilon')^{-\Delta}, \qquad \Pi_m(\sigma) = \frac{C^{(0)}_m}{F^{(\Delta)}_m} \prod_{r=1}^{\Delta} \frac{C^{(0)}_{m-r} D^{(1)}_{m-r}}{F^{(\Delta)}_{m-r} F'^{(-\Delta)}_{m-r}}, \tag{5·6}
$$

$$
\text{case C}: \quad r_E = \frac{y^{\Delta}_E}{y_E} \frac{\langle \omega'_{-\Delta} \rangle}{\langle \phi'_0 \rangle} (\epsilon')^{-\Delta}, \qquad \Pi_m(\sigma) = \frac{F'^{(-\Delta)}_m}{D^{(0)}_m} \prod_{r=1}^{\Delta} \frac{F^{(\Delta+1)}_{m-r} F'^{(-\Delta)}_{m-r}}{C^{(0)}_{m-r} D^{(0)}_{m-r}}, \tag{5·7}
$$

$$
\text{case D}: \quad r_E = \frac{y^{\Delta}_E}{y_E} \frac{\langle \psi'_0 \rangle}{\langle \chi'_{\Delta} \rangle} (\epsilon')^{-\Delta}, \qquad \Pi_m(\sigma) = \frac{C^{(0)}_m}{F^{(\Delta)}_m} \prod_{r=1}^{\Delta} \frac{F^{(\Delta)}_{m-r+1} F'^{(1-\Delta)}_{m-r}}{C^{(0)}_{m-r+1} D^{(0)}_{m-r}}. \tag{5·8}
$$

The expression of ϵ' is given in (4·22) for each case. To preserve the hierarchy $y^{mn}_e \propto \epsilon'^{m+n}$ in the coupling matrix (5·4), r_E must be $O(1)$. This requires us to assign the orders of the VEVs to

$$
\text{case A, C}: \quad \langle \omega'_{-\Delta} \rangle \simeq \epsilon'^{\Delta} \langle \phi'_0 \rangle, \qquad \text{case B, D}: \quad \langle \psi'_0 \rangle \simeq \epsilon'^{\Delta} \langle \chi'_{\Delta} \rangle. \tag{5·9}
$$

The coupling matrix y_d^{mn} for down-type quarks is obtained from (5·4) by the adequate replacement;

$$y_d^{mn} = y_D C^D_{m,n} \left(1 - r_D \theta_{m+n,\Delta} \frac{C^{D\Delta}_{m,n}}{C^D_{m,n}} \Pi_{m+n}(\sigma) \right) U'_{m+n}, \tag{5·10}$$

where

$$C^D_{m,n} = C^{\gamma,\alpha}_{m,n}(0), \qquad C^{D\Delta}_{m,n} = C^{\gamma,\alpha}_{m,n}(\Delta), \qquad r_D = r_E \frac{y_E y_D^\Delta}{y_D y_E^\Delta}. \tag{5·11}$$

If we are modest and hesitate to introduce the multiple hierarchies among the VEVs, we may be led to take $\Delta = 0$ or $\Delta = 1$. When $\Delta = 0$, we recognize case B and case C are equivalent to case A and case D, respectively, under the replacement of Ψ' and Φ', because H' and K' now belong to the same representation. In these cases, the coupling matrix y_e^{mn} takes a simple form

$$\text{Type-I-A}_{\Delta=0}: \qquad y_e^{mn} = y_E C^E_{m,n} \left(1 - r_E \frac{F'^{(0)}_{m+n}}{D^{(0)}_{m+n}} \right) U'_{m+n}, \tag{5·12}$$

$$\text{Type-I-D}_{\Delta=0}: \qquad y_e^{mn} = y_E C^E_{m,n} \left(1 - r_E \frac{C^{(0)}_{m+n}}{F^{(0)}_{m+n}} \right) U'_{m+n}. \tag{5·13}$$

When $\Delta = 1$, we see the VEVs of case B requires the multiple hierarchies among the VEVs; $\langle \omega'_{-1} \rangle \simeq \epsilon'^2 \langle \phi'_1 \rangle$, $\langle \psi'_0 \rangle \simeq \epsilon' \langle \chi'_1 \rangle$. The situation is the same for case C. The VEVs of case A and case D give

$$\text{Type-I-A}_{\Delta=1}: \qquad y_e^{mn} = y_E C^E_{m,n} \left(1 - r_E \theta_{m+n,1} \frac{C^{E\Delta}_{m,n} C^{(1)}_{m+n-1}}{C^E_{m,n} F^{(1)}_{m+n-1}} \right) U'_{m+n}, \tag{5·14}$$

$$\text{Type-I-D}_{\Delta=1}: \qquad y_e^{mn} = y_E C^E_{m,n} \left(1 - r_E \theta_{m+n,1} \frac{C^{E\Delta}_{m,n} F'^{(0)}_{m+n-1}}{C^E_{m,n} D^{(0)}_{m+n-1}} \right) U'_{m+n}. \tag{5·15}$$

Type-II scheme

The Yukawa coupling matrices of the Type-II scheme have a simple form

$$y_e^{mn} = y_E C^E_{m,n} U'_{m+n}, \qquad y_d^{mn} = y_D C^D_{m,n} V'_{m+n}, \tag{5·16}$$

with C-G coefficients given in (5·3) and (5·11). The weights are now restricted by (4·7). The natural hierarchy of the coupling matrices is realized by the recursion equations (4·18)∼(4·21). The input values U'_0 and V'_0 are connected by the definite relation

$$\text{case A, C}: \quad V'_0 = -\frac{\langle \omega'_{-\Delta} \rangle}{\langle \phi'_0 \rangle} \frac{F'^{(-\Delta)}_0}{D^{(0)}_0} U'_0, \quad \text{case B, D}: \quad V'_0 = -\frac{\langle \psi'_0 \rangle}{\langle \chi'_\Delta \rangle} \frac{C^{(0)}_0}{F^{(\Delta)}_0} U'_0. \tag{5·17}$$

Characteristics of the coupling matrix

Let us first point out one important fact that concerns both Type-I and Type-II schemes. The expression of ϵ', which is concisely represented in (4·22), shows that ϵ' transforms under the $U(1)_H$ subgroup of $SU(1,1)$ as if it is a "VEV with weight -1" in all cases A∼D, that precisely coincides with (3·14) of the minimal model. This is not an accident but a consequence of the requirement that only one down-type higgs doublet h' is realized as a chiral superfield. This implies that what has been meant in terms of the vague phrase "natural hierarchy" is not a smallness of ϵ' but its definite transformation property under $U(1)_H$.[2),24)] The magnitude of the hierarchy is a consequence of the non-Abelian group structure of $SU(1,1)$.

There are several points which should be mentioned on the property of the Type-I scheme. First of all, this scheme admits two extra free parameters r_E and r_D. They are in general complex numbers, but we realize, from their relation (5·11), they have common phase because all of the coupling constants y_E, y_E^{Δ}, y_D, y_D^{Δ} are real numbers under the "fundamental principle" of P-C-T-invariance. This phase should be a physical one independent of the phase convention of the VEVs. This is confirmed from its expressions (5·5)∼(5·8), that insure r_E to have "weight 0" and do not move under $U(1)_H$ rotation. Since we have no explicitly complex number in the basic framework, it will be natural to expect

$$\frac{r_E}{|r_E|} = \pm\frac{r_D}{|r_D|} = e^{i\pi q/p} \tag{5·18}$$

with some set of integers p and q. The simplest candidate is of course $q/p =$ [integer], that is, r_E and r_D are pure real (Option-1). This phase assignment brings a fascinating chance for us to have a texture-zero[21)–23)] in y_e^{mn} and y_d^{mn} at specific values of r_E and r_D by the cancellation of two terms in the coupling matrix. This texture-zero appears in a remarkable pattern in $\Delta = 0$ scheme. From their expressions (5·12) and (5·13), we find all of the matrix elements with common $m + n$ vanish together. It should, however, be sensible to expect this cancellation occurs, at most, approximately rather than exactly, because we have no principle in the framework that guarantees the exact cancellation. The uncomfortable aspect of the Option-1 phase assignment is that it causes the instability of the natural hierarchy of coupling matrices at various values of r_E and r_D through the accidental cancellation in the eigenvalues of the coupling matrix. That is, would-be $O(1)$ eigenvalue becomes $O(\epsilon'^2)$ and would-be $O(\epsilon'^2)$ eigenvalue becomes $O(\epsilon'^4)$. Such a phenomenon does not occur in the minimal $SU(1,1)$ model when the marginal assignment $S_0 = S_1 \equiv S$ is taken in (3·12); the condition $X(\sigma) = 0$ in (3·26) leads to inaccessible σ-independent constraint $S^2 + S = 0$. The optimum that maximally stabilizes the hierarchy in Type-I scheme will be $q/p =$ [half-interger], that is, r_E and r_D are pure imaginary (Option-2).

The second point is on the phase of ϵ' which is also complex in general. However, this phase is dependent on the phase convention of the VEVs because it has "weight -1". Therefore we can always rotate out the phase of ϵ' so that it becomes real and positive. This does not mean at all that this phase is completely unphysical. When we have another quantity ϵ which coherently moves with ϵ' under $U(1)_H$ rotation, the relative phase of ϵ' and ϵ is a physical observable. This ϵ surely exists in our framework as a mixing parameter in the up-type higgs sector, which we are not discussing in this paper. This relative phase and the phase of r_E and r_D are reflected on the phases in the CKM and MNS matrices.

The third point on the Type-I scheme, which may be the most appealing point, is that this scheme allows us to assign all quarks and leptons to the common $SU(1,1)$ representation. When this assignment is adopted in $\mathit{\Delta} =$ [odd number] schemes, the first part in the coupling matrices (5·4) and (5·10) becomes a symmetric matrix but the second part which contains r's becomes antisymmetric owing to the symmetry relation $C^{\lambda,\eta}_{i,j}(\mathit{\Delta}) = (-1)^{\mathit{\Delta}} C^{\eta,\lambda}_{j,i}(\mathit{\Delta})$ of (3·6). This means that the Option-2 phase assignment realizes the hermitian Yukawa coupling matrices[23)] under the real ϵ' phase convention. On the other hand, $\mathit{\Delta} =$ [even number] schemes give perfectly symmetric coupling matrices.

The last point which should be mentioned on the Type-I scheme is the assumption in (4·4). In principle, there is a possibility that K' takes the lowest coupling, for example, in $\bar{D}QK'$. In this case, $\bar{D}QH'$ is forbidden and y_d^{mn} takes simpler form without parameter r_D. When both of $\bar{E}LK'$ and $\bar{D}QK'$ take the lowest coupling, the situation reduces to the minimal model, that has been ruled out.

The typical property of the Type-II scheme is that, under the real ϵ' phase convention, y_e^{mn} and y_d^{mn} become pure real matrices up to overall phases connected by (5·17). This means that, when the up-type higgs doublet h also takes the Type-II mixing scheme (or Type-I scheme with Option-1 phase assignment), the origin of the phase of the CKM matrix is solely the relative phase of ϵ and ϵ', $\delta \equiv \arg(\epsilon/\epsilon')$. This restricts the form of the CKM matrix to

$$V_{\rm CKM} = O_u^T P O_d, \qquad P \equiv \begin{pmatrix} e^{2i\delta} & 0 & 0 \\ 0 & e^{i\delta} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{5·19}$$

where O_u and O_d are the real orthogonal matrices, standing on the right when the "reversed" $(m, n = 2, 1, 0)$ coupling matrices y_u^{mn} and y_d^{mn} with real and positive ϵ and ϵ' are diagonalized, respectively. The analysis of Type-II scheme will be an exciting subject because we have no free parameter like r_E and r_D. All mass ratios must be reproduced in terms of ϵ' and the weights of each multiplet, that we failed in the minimal $SU(1,1)$ model.

§6. Summary and discussions

The simplest non-Abelian noncompact group $SU(1,1)$ gives us an excellent viewpoint on what is realized in the low energy physics when it is introduced, in the framework of the vector-like theory, as a symmetry group of the generations of quarks and leptons. It gives, in terms of its spontaneous breaking, an answer to the questions why three generations of quarks and leptons are simply repeated and why they have, never-the-less, rich hierarchical mass structures.

In this paper we first investigated the structure of the Yukawa coupling matrices y_e^{mn} and y_d^{mn} of the minimal $SU(1,1)$ model, which contains single down-type higgs multiplet H', and found this minimal model gives a definite prediction on the mass ratios, that is phenomenologically far unacceptable. If Nature really uses $SU(1,1)$, this result means that the higgs sector should have richer structure than that of the minimal model. Following this observation, we formulated the minimal extension of the higgs sector by incorporating additional higgs multiplet K'.

The extension of the higgs sector is a non-trivial subject because the success of MSSM insists that there is only one down-type higgs doublet h' at low energy. This requires the special mixing scheme that realizes single h' at low energy from doubled higgs sector which consists of H' and K' (and their conjugates). We found throughout the formulation that the general framework of the model admits this mixing scheme in quite restrictive way. As a result, it was shown that the possible forms of the Yukawa couplings are classified to two types, Type-I and Type-II, each of which has four patterns of mixing, A, B, C, and D. Each scheme reveals its own property in the specific structure of the coupling matrices.

Although the principle is simple, the resulting expressions of the coupling matrices are much complicated. It is difficult, at present, to make definite statement on which type of scheme is phenomenologically most preferable. The decision should wait the result of the global analysis of the full scheme of the model that contains the mixing scheme also for the up-type higgs h. It should be mentioned, however, that the preliminary numerical analysis on the Type-I scheme shows there are some sets of parameters that reproduce the mass ratios reasonably. In the following, we give the typical examples. All mass ratios should be compared with the ratios at the GUT-scale.[18] In each case, the marginal assignment (4·29) is adopted.

In the analysis, we assumed a universal quark and lepton assignment

$$2\alpha = 2\gamma = 2\eta = 2\lambda = \sigma \tag{6·1}$$

within the restricted range of the discrete values $\sigma = 1/2, 1, 3/2, 2, 5/2, 3, 7/2$. We found the

Type-I-A$_{\varDelta=1}$ scheme with $S_\varPsi = S_\varPhi = S_X = 3$, which requires $|\epsilon'| < L = 2.0_5$, accepts the set of parameters

$$\left\{ \begin{array}{ll} \sigma = 1, & \epsilon' = 0.77 \\ r_E = \pm 1.33, & r_D = \pm 0.41 \end{array} \right. \text{gives} \left\{ \begin{array}{ll} m_\mu/m_\tau = 0.060_0, & m_e/m_\mu = 0.0048_8 \\ m_s/m_b = 0.017_0, & m_d/m_s = 0.044_5 \end{array} \right. . \qquad (6\cdot 2)$$

Also in the Type-I-D$_{\varDelta=0}$ scheme with $S_\varPsi = 2$, $S_\varPhi = 0$, $S_X = 1$, which requires $|\epsilon'| < L = 1.0_6$,

$$\left\{ \begin{array}{ll} \sigma = 3/2, & \epsilon' = 0.72 \\ r_E = 0.94, & r_D = 1.38 \end{array} \right. \text{gives} \left\{ \begin{array}{ll} m_\mu/m_\tau = 0.057_0, & m_e/m_\mu = 0.0048_2 \\ m_s/m_b = 0.020_1, & m_d/m_s = 0.053_1 \end{array} \right. . \qquad (6\cdot 3)$$

The parameter search for the Type-II scheme is now under investigation.

We would like to conclude this paper with a briefly discussion on the "full set" of the superpotential W of the model. For definiteness, we assume the Type-I$_{\varDelta>0}$ mixing scheme for both of up-type and down-type higgs sectors. W consists of the four parts;

$$W = W_{\mathrm{MSSM}} + W_{\mathrm{N}} + W_{\mathrm{M}} + W(\text{finite dim.}). \qquad (6\cdot 4)$$

The first part W_{MSSM} reproduces MSSM up to μ-term;

$$\begin{aligned} W_{\mathrm{MSSM}} = & (Q\bar{Q} + \bar{U}U + \bar{D}D + L\bar{L} + \bar{E}E)(\varPsi^F + \varPsi'^F) \\ & + H\bar{H}\varPsi + K\bar{K}\varPhi + K\bar{H}X + H\bar{K}\varOmega \\ & + H'\bar{H}'\varPsi' + K'\bar{K}'\varPhi' + K'\bar{H}'X' + H'\bar{K}'\varOmega' \\ & + \bar{U}Q(H+K) + \bar{D}Q(H'+K') + \bar{E}L(H'+K') + QQD + Q\bar{U}\bar{L} \\ & + U\bar{Q}(\bar{H}+\bar{K}) + D\bar{Q}(\bar{H}'+\bar{K}') + E\bar{L}(\bar{H}'+\bar{K}') + \bar{Q}\bar{Q}\bar{D} + \bar{Q}UL. \end{aligned} \qquad (6\cdot 5)$$

We assume H and H' take the lowest coupling in the operators in the fourth line.

The second part W_{N} is responsible to the neutrino masses. Since the Majorana mass operator of a type $NN\varPsi^N$ is forbidden by the $SU(1,1)$ symmetry, the see-saw mechanism[25] does not work so efficiently in the present framework. The adequate alternative is to introduce SU_2 triplets.[9] The observed large mixing angles[26)–28] in the MNS matrix seem to require some sets of triplets T^i and their conjugates;

$$\begin{aligned} W_{\mathrm{N}} = & \sum_{i,j} T^i \bar{T}^j \varPsi^{ij} + LL \sum_i \bar{T}^i + (HH + HK + KK) \sum_i T^i \\ & + \bar{L}\bar{L} \sum_i T^i + (\bar{H}\bar{H} + \bar{H}\bar{K} + \bar{K}\bar{K}) \sum_i \bar{T}^i. \end{aligned} \qquad (6\cdot 6)$$

We must be careful so that the VEVs $\langle \varPsi^{ij} \rangle$ do not generate massless particles in T^i and $\bar{T}^i$. Then, the integration of T^i and $\bar{T}^i$ produces the neutrino mass operator

$$\sum_{m,n=0}^{2} \kappa_\nu^{mn} \ell_m h \ell_n h \qquad (6\cdot 7)$$

with the coupling matrix $\kappa_\nu^{mn} \simeq O(M^{-1})$. The structure of κ_ν^{mn} is sensitive to the assignment of the weights of each multiplet and the pattern of the VEVs $\langle \Psi^{ij} \rangle$.

The third part $W_{\rm M}$ contains SU_2 singlets $R_{(\rho+\sigma)/2}$, $R'_{(\rho+\sigma)/2+1}$, $S_{\rho+\sigma}$ and their conjugates. It is responsible to the μ-term $\mu hh'$;

$$W_{\rm M} = R\bar{R}'\Psi^R + R'\bar{R}\Psi'^R + R\bar{R}'\Psi^M + R'\bar{R}\Psi'^M + S\bar{S}\Psi^S \\ +HH'S + RR\bar{S} + \bar{H}\bar{H}'\bar{S} + \bar{R}\bar{R}S. \tag{6·8}$$

The VEVs $\langle \psi_1^R \rangle$, $\langle \psi'^R_{-1} \rangle$, $\langle \psi_0^M \rangle$, $\langle \psi'^M_0 \rangle$, $\langle \psi_0^S \rangle$ generate a set of massless r and $\bar{r}$ in R and $\bar{R}$, respectively, to which the superheavy $\bar{S}$ and S couple. When the supersymmetry breaking terms characterized by the scale $m_{\rm SUSY} \simeq 10^{2\sim3}$GeV are incorporated in the scheme, r and $\bar{r}$ acquire the VEVs in the intermediate scale $\sqrt{Mm_{\rm SUSY}}$. The integration of S and $\bar{S}$ eventually induces the μ-term at the scale $\mu \simeq m_{\rm SUSY}$. At this time, we must take care in the assignment of the weight of L so that the dangerous couplings $L(H+K)R$ and $L(H+K)\bar{R}$ are forbidden by the weight constraint.

The resulting mass spectrum in the low energy effective theory contains, in addition to the MSSM particles, several neutral particles coming from r and $\bar{r}$. Most of them have masses of order $m_{\rm SUSY}$, but their couplings to higgses (hh') are suppressed by the factor $\sqrt{m_{\rm SUSY}/M}$. What is unexpected is the emergence of the long-range force mediated by the "exactly massless" pseudo-scalar Nambu-Goldstone (N-G) boson G^0, which originates from the spontaneous breakdown of the Peccei-Quinn $U(1)_{\rm PQ}$ symmetry.[29)]

The appearance of the $U(1)_{\rm PQ}$ symmetry is a consequence of the $SU(1,1)$ symmetry, which restricts the possible couplings of matter multiplets by the weight constraint. The finite-dimensional multiplets will not share the $U(1)_{\rm PQ}$ charge since the $SU(1,1)$ symmetry does not impose so stringent constraint on their couplings in W(finite dim.). The N-G boson G^0 couples to quarks and leptons through the mixing with the pseudo-scalar particle a^0 in the MSSM with the mixing fraction of order $\sqrt{m_{\rm SUSY}/M}$. At a glance, it seems to be almost the invisible axion.[30)] The essential difference is now the $U(1)_{\rm PQ}$ symmetry is an "exact" symmetry free from gauge anomalies because the basic framework of the model is purely vector-like. Therefore, G^0 does not couple to two photons nor two gluons through gauge anomalies.

In principle, we cannot deny a possibility that the $U(1)_{\rm PQ}$ symmetry is explicitly broken in the couplings of the matter multiplets to the superheavy multiplets $(Z^d, \bar{Z}^d)$ which we have been discarding. In this case, the $U(1)_{\rm PQ}$ symmetry revives in the low energy effective theory as if it is an "anomalous" symmetry. When this is the case, G^0 will acquire a mass suppressed by the huge mass M of the superheavy multiplets. So, it will be prudent to imagine G^0 has a tiny mass $m_G \simeq (m_{\rm SUSY})^{n+1}/M^n$ with some positive integer n. Not only

the $U(1)_{\rm PQ}$ symmetry, but also the baryon number $U(1)_B$ and the lepton number $U(1)_L$ symmetries, though explicitly broken, do not suffer from gauge anomalies. These results may give significant impacts on the understanding of the present universe.

Finally, the fact that the $U(1)_{\rm PQ}$ symmetry does not suffer from gauge anomalies means that G^0 loses the role as a solution to the strong CP problem.[31] As an alternative, we have now a exact P-C-T-invariance[32] at the fundamental level. Therefore we may have a chance to solve the problem based on this invariance supplemented by the $U(1)_{\rm PQ}$ symmetry. We would like to leave this subject to the future study.

To be honest, we should state that we do not have, at present, a well established background that allows us to make such an unconventional scenario. We are anticipating the off-shell property of the superstring theory, when fully clarified, gives a reliable support.

Acknowledgments

The authors would like to thank K. Yoshioka for enlightening suggestion, K. Harada for stimulating encouragement and A. Watanabe, K. Kojima, H. Sawanaka and X. Tata for helpful discussions. This work is supported in part by the Grant-in-aid for the scientific research on priority area (♯ 441) "Progress in elementary particle physics of the 21st century through discoveries of Higgs boson and supersymmetry" (No.16081209) from the Ministry of Education, Science, Sports, and Culture of Japan.

References

1) T. Maehara and T. Yanagida, Prog. Theor. Phys. **61** (1979), 1434.
F. Wilczek and A. Zee, Phys. Rev. Lett. **42** (1979), 421.

2) C. D. Froggatt and H. B. Nielsen, Nucl. Phys. B **147** (1979), 277.

3) K. Inoue, Prog. Theor. Phys. **93** (1995), 403; Prog. Theor. Phys. Suppl. No. 123 (1996), 319.

4) F. A. Wilczek, A. Zee, R. L. Kingsley and S. B. Treiman, Phys. Rev. D **12** (1975), 2768.
A. De. Rújula, H. Georgi and S. L. Glashow, Phys. Rev. D **12** (1975), 3589.
H. Fritzsch, M. Gell-Mann and P. Minkowski, Phys. Lett. B **59** (1975), 256.
K. Inoue, A. Kakuto and Y. Nakano, Prog. Theor. Phys. **58** (1977), 630.
M. Yoshimura, Prog. Theor. Phys. **58** (1977), 972.

5) K. Fujikawa, J. Korean Phys. Soc. **29** (1996), S247; Prog. Theor. Phys. **92** (1994), 1149.

6) P. Fayet, Phys. Lett. B **69** (1977), 489.

K. Inoue, A. Kakuto, H. Komatsu and S. Takesita, Prog. Theor. Phys. **67** (1982), 1889; ibid. **68** (1982), 927.

L. E. Ibáñez and G. G. Ross, Phys. Lett. B **110** (1982), 215.

L. Alvarez-Gaume, J. Polchinski and M. K. Wise, Nucl. Phys. B **221** (1983), 495.

J. Ellis, J. S. Hagelin, D. V. Nanopoulos and K. Tamvakis, Phys. Lett. B **125** (1983), 275.

7) D. J. H. Chung, et.al., Phys. Rep. **407** (2005), 1.

S. P. Martin, hep-ph/9709356.

8) K. Inoue and N. Yamashita, Prog. Theor. Phys. **104** (2000), 677.

9) K. Inoue and N. Yamashita, Prog. Theor. Phys. **110** (2003), 1087.

10) J.R. Ellis, S. Kelley and D.V. Nanopoulos, Phys. Lett. B **260** (1991), 131.

U. Amaldi, W. de Boer and H. Furstenau, Phys. Lett. B **260** (1991), 447.

P. Langacker and M.x. Luo, Phys. Rev. D **44** (1991), 817.

11) S. Weinberg, Phys. Rev. D **26** (1982), 287.

N. Sakai and T. Yanagida, Nucl. Phys. B **197** (1982), 533.

12) R. N. Mohapatra, et al., Repot. Prog. Phys. **70** (2007), 1757.

13) A. Strumia and F. Vissani, hep-ph/0606054.

14) N. Cabibbo, Phys. Rev. Lett. **10** (1963), 531.

M. Kobayashi and K. Maskawa, Prog. Theor. Phys. **49** (1972), 282.

15) Z. Maki, M. Nakagawa and S. Sakata, Prog. Theor. Phys. **28** (1962), 870.

16) W. M. Yao et al. (Particle Data Group), J. Phys. G **33**, (2006), 1.

17) H. Georgi and C. Jarlskog, Phys. Lett. B **86** (1979), 297.

18) G. Ross and M. Serna, arXiv:0704.1248 [hep-ph].

19) H. Georgi and S. L. Glashow, Phys. Lett. **32** (1974), 438.

20) E. Witten, Nucl. Phys. B **188** (1981), 513.

N. Sakai, Z. Phys. **C 11** (1981), 153.

S. Dimopoulos and H. Georgi, Nucl. Phys. B **193** (1981), 150.

21) H. Fritzsch, Phys. Lett. B **70** (1977), 436; ibid. **73** (1978), 317.

22) P. Kaus and S. Meshkov, Mod. Phys. Lett. **A3** (1988), 1251; **A 4** (1989), 603.

F.J. Gilman and Y. Nir Ann, Rev. Nucl. Part. Sci. **40** (1990), 213.

N. Uekusa, A. Watanabe and K. Yoshioka, Phys. Rev. D **71** (2005), 094024.

A. Watanabe and K. Yoshioka, J. High Energy Phys. **0605** (2006), 044.

N. Haba, A. Watanabe and K. Yoshioka, Phys. Rev. Lett. **97** (2006), 041601.

S. Kaneko, H. Sawanaka, T. Shingai, M. Tanimoto and K. Yoshioka, Prog. Theor. Phys. **117** (2007), 161.

23) P. Ramond, R.G. Roberts and G.G. Ross, Nucl. Phys. B **406** (1993), 19.

24) N. Maekawa, Prog. Theor. Phys. **106** (2001), 401.

25) T. Yanagida, in *Proceedings of the Workshop on the Unified Theory and Baryon Number in the Universe*, ed. O. Sawada and A. Sugamoto (KEK report 79-18, 1979). M. Gell-Mann, P. Ramond and R. Slansky, in *Supergravity*, ed. P. van Nieuwenhuizen and D. Z. Freedman (North Holland, Amsterdam, 1979).

26) Y. Fukuda et al., Phys. Lett. B **433** (1998), 9; ibid. **436** (1998), 33; Phys. Rev. Lett. **81** (1998), 1562; ibid. **82** (1999), 2644; ibid. **93** (2004), 101801.

27) S. Fukuda et al., Phys. Rev. Lett. **86** (2001), 5651; Phys. Lett. B **539** (2002), 179. Q.R. Ahmad et al., Phys. Rev. Lett. **89** (2002), 011301; ibid. **89** (2002), 011302.

28) K. Eguchi et al., Phys. Rev. Lett. **90** (2003), 021802.

29) R. D. Peccei and H.R. Quinn, Phys. Rev. Lett. **38** (1977), 1440.

30) J.E. Kim, Phys. Rev. Lett. **43** (1979), 103.
M.A. Shifman, A.I. Vainstein and V.I. Zakharov, Nucl. Phys. B **166** (1980), 493.
A.R. Zhitnitsky, Sov. J. Nucl. Phys. **31** (1980), 260.
M. Dine and W. Fischler, Phys. Lett. B **120** (1983), 127.

31) A.A. Belavin, A.M. Polyakov, A.S. Schwartz and Y.S. Tyupkin, Phys. Lett. B **59** (1975), 85.
G. 't Hooft, Phys. Rev. Lett. **37** (1976), 8; Phys. Rev. D **14** (1976), 3432.
R. Jackiw and C. Rebbi, Phys. Rev. Lett. **37** (1976), 172.
C.G. Callan, R.F. Dashen and D.J. Gross, Phys. Lett. B **63** (1976), 334.

32) S.M. Barr and A. Zee, Phys. Rev. Lett. **55** (1985), 2253.